Band structures and band gaps of liquid surface waves propagating through an infinite array of cylinders

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The multiple scattering method is applied to the calculations of band structures of liquid surface waves propagating through an infinite array of vertical cylinders. The influence of the filling fraction on the formation of band gaps is discussed. It is found that there exist complete band gaps for both the square and triangular arrays of cylinders.

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In the last decade there has been considerable interest in photonic crystals owing to their interesting physical properties and potential applications in novel devices as well [1–6]. As a result of multiple Bragg scatterings, the propagation of electromagnetic waves is characterized by photonic band structures. Between photonic bands, there may exist a photonic band gap, within which wave propagation is absolutely forbidden.

When propagating in periodic structures, liquid surface waves will also be modulated by the introduced periodicity. Band structures and band gaps can also exist for liquid surface waves. As a result, many interesting phenomena found in photonic crystals may also exist in liquid surface waves. One of the advantages by using liquid surface waves, from the experimental point of view, is that these interesting phenomena can be observed directly by visualizing the patterns of liquid surface waves. For example, Bloch waves and domain walls have been visualized for liquid surface waves propagating over a bottom with periodically drilled holes [7].

Recently, there have been some theoretical [8–11] and experimental studies [11] on the band structures and the possibility of the existence of band gaps for liquid surface waves propagating in periodic structures. By using the plane-wave expansion method, Chou [9] studied the band structures of surface water waves along periodic interfaces. In two-dimensional periodic geometries, no complete band gaps were found. Torres *et al.* [11] investigated theoretically and experimentally the band structures of liquid surface waves over a bottom with periodically drilled two-dimensional holes. In their band structure calculations, the plane-wave expansion method was used. But no complete band gaps were observed. By using a variational method, McIver [10] studied water wave propagation through an array of vertical cylinders arranged in the square lattice. But only the band structure along one direction in the Brillouin zone was given.

In photonic crystals, the plane-wave expansion method has been commonly used in the calculations of photonic band structures [12–14]. But the multiple scattering method has also been used in the calculations of transmission properties and photonic band structures as well [15,16]. For liquid surface waves, the multiple scattering method, which is particularly efficient for arrays of axisymmetric bodies, has been frequently adopted in the study of many properties of propagation of liquid surface waves [17–22]. To our knowledge, however, there are no calculations of the band structures of liquid surface waves based on the multiple scattering method.

In this paper we apply the multiple scattering method to calculate the band structures of liquid surface waves propagating through an infinite two-dimensional array of vertical cylinders. Consider a periodic array of identical, rigid, circular cylinders standing in liquid of constant depth h. The liquid is assumed to be inviscid and incompressible and the flow to be irrotational. Set $x-y$ in the horizontal plane and z as the vertical axis. The free surface of the calm liquid is at $z=0$ and the bottom at $z=-h$. On the basis of inviscid, linear theory of liquid surface waves [23,24], the velocity potential Φ may be sought in the form

$$\Phi(x,y,z,t)=\mathrm{Re}[\phi(x,y)\cosh\kappa(z+h)e^{-i\omega t}], \tag{1}$$

where ω is the angular frequency of a harmonic mode. ϕ is given by the solutions of the two-dimensional Helmholtz equation

$$(\nabla^2+\kappa^2)\phi=0, \tag{2}$$

which is subjected to the boundary condition of no flow through the cylinder walls, namely,

$$\frac{\partial\phi}{\partial\mathbf{n}}=0, \tag{3}$$

where $\mathbf{n}$ is the direction normal to the cylinder surface. The wave number κ can be obtained from the dispersion relation [23,24]

$$\omega^2=g\kappa\tanh\kappa h, \tag{4}$$

where g is the gravitational acceleration. The vertical displacement of the liquid surface η is related to ϕ by [23,24]

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$$\eta(x,y,t)=\mathrm{Re}\left[-\frac{i\omega}{g}\phi(x,y)e^{-i\omega t}\right]. \tag{5}$$

Equation (2) can be solved by the multiple scattering method. For the infinite system, however, there is a computational difficulty due to the slow convergence of the lattice sums. Fortunately, this difficulty can be overcome by a new technique, introduced in photonic crystals [25,26], to accelerate the convergence. It should be mentioned that the problem of liquid surface waves propagating through an array of vertical cylinders is very similar to that of electromagnetic waves with p polarization propagating in an array of perfectly conducting cylinders. Therefore, the techniques used in electromagnetic waves [26] can be adopted in the present work. The multiple scattering method used to calculate band structures of liquid surface waves is briefly summarized below.

For a finite system, a source must be introduced. But for band structure calculations it is not necessary to introduce a source. Without a source term, the field ϕ at any point of liquid area is composed of the scattering cylindrical waves from all the cylinders

$$\phi(\mathbf{r})=\sum_{j=1}^{N}\sum_{m=-M}^{M}B_{j,m}H_m(\kappa|\mathbf{r}-\mathbf{r}_j|)e^{im\arg(\mathbf{r}-\mathbf{r}_j)}, \tag{6}$$

where H is the first Hankel function, m is the index of order, and j is the index of cylinders. In the vicinity of the cylinder j, the field ϕ can be then written as the composition of the incident and outgoing (scattering) waves

$$\phi(\mathbf{r})=\sum_{m=-M}^{M}[A_{j,m}J_m(\kappa|\mathbf{r}-\mathbf{r}_j|)+B_{j,m}H_m \times(\kappa|\mathbf{r}-\mathbf{r}_j|)]e^{im\arg(\mathbf{r}-\mathbf{r}_j)}, \tag{7}$$

where the first and the second terms denote the incident and outgoing waves, respectively, and J is the Bessel function of the first kind. The ratio of the incident to the scattering coefficients can be determined from Eq. (3), as

$$D_m\equiv\frac{B_{j,m}}{A_{j,m}}=-\frac{J'_m(\kappa R)}{H'_m(\kappa R)}, \tag{8}$$

where R is the radius of cylinders, and the order m of cylindrical waves.

The geometry description of an infinite array (or lattice) of cylinders adopted is that used in solid state physics [27]. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be the two noncollinear primitive translation vectors of the lattice. The intersections of cylinders form a two-dimensional Bravais lattice whose sites are given by

$$\mathbf{r}_l=l_1\mathbf{a}_1+l_2\mathbf{a}_2, \tag{9}$$

where l_1 and l_2 are any two integers which are denoted collectively by l. It is convenient to introduce so-called reciprocal lattice vectors

$$\mathbf{G}_h=h_1\mathbf{b}_1+h_2\mathbf{b}_2, \tag{10}$$

where h_1 and h_2 are integers which are denoted collectively by h and the primitive translation vectors of the reciprocal lattice are given by

$$\mathbf{b}_1=\frac{2\pi}{a_c}(a_2^{(2)},-a_1^{(2)}), \tag{11}$$

$$\mathbf{b}_2=\frac{2\pi}{a_c}(-a_2^{(1)},-a_1^{(1)}), \tag{12}$$

where $a_j^{(i)}$ is the jth Cartesian component of $\mathbf{a}_i$ and $a_c=|\mathbf{a}_1\times\mathbf{a}_2|$ is the area of the primitive unit cell.

Within the framework of the Bloch theorem, the solutions of Eq. (2) should be Bloch waves. The incident and scattering waves of the cylinder j are associated with the center cylinder ($\mathbf{r}=\mathbf{0}$) by a phase factor $e^{i\mathbf{K}\cdot\mathbf{r}_j}$, namely,

$$A_{j,m}=A_m e^{i\mathbf{K}\cdot\mathbf{r}_j}, \tag{13}$$

$$B_{j,m}=B_m e^{i\mathbf{K}\cdot\mathbf{r}_j}, \tag{14}$$

where A_m and B_m are for the central cylinder and $\mathbf{K}$ is the Bloch wave vector, which lies in the first Brillouin zone [27]. By means of Eqs. (6) and (13), following the method developed by Lord Rayleigh [28], the following linear equations for B_m can be obtained:

$$\sum_{m=-M}^{M}Q_{n,m}(\kappa,\mathbf{K})B_m=0, \tag{15}$$

where

$$Q_{n,m}(\kappa,\mathbf{K})=-\frac{1}{D_n}\delta_{n,m}+S_{n-m}^A(\kappa,\mathbf{K}).$$

The coefficients S_{n-m}^A are lattice sums given by the formula [25,26]

$$S_l^A(\kappa,\mathbf{K})=-\delta_{l,0}+iS_l^Y(\kappa,\mathbf{K}). \tag{16}$$

If $l\geqslant 0$,

$$S_l^Y(\kappa,\mathbf{K})J_{l+m}(\kappa)=-\left[Y_m(\kappa)+\frac{1}{\pi}\times\sum_{n=1}^{m}\frac{(m-n)!}{(n-1)!}\left(\frac{2}{\kappa}\right)^{m-2n+2}\right]\delta_{l,0}-i^l\frac{4}{A}\sum_h\left(\frac{\kappa}{|\mathbf{Q}_h|}\right)^m\frac{J_{l+m}(|\mathbf{Q}_h|)}{|\mathbf{Q}_h|^2-\kappa^2}e^{il\arg(\mathbf{Q}_h)}, \tag{17}$$

and if $l<0$,

$$S_l^Y(\kappa,\mathbf{K})=[S_{-l}^Y(\kappa,\mathbf{K})]^*, \tag{18}$$

where Y is the Bessel function of the second kind and $\mathbf{Q}_h=\mathbf{K}+\mathbf{G}_h$. To ensure Eq. (15) has nontrivial solutions, the determinant of the matrix $Q(\kappa,\mathbf{K})$ must be zero

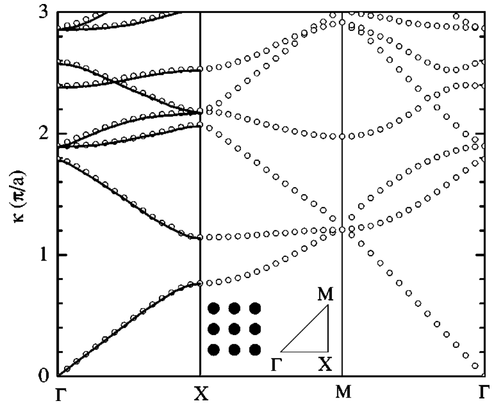

FIG. 1. Wave number κ versus wave vectors in the Brillouin zone along three highly symmetrical directions for the square. The filling fraction is 0.196. Solid lines denote the results of McIver [10] and dots denote our calculated results. The geometry of cylinders (top view) and the irreducible Brillouin zone are shown as insets.

$$\det|Q(\kappa,\mathbf{K})|=0. \tag{19}$$

Band structures can be then obtained by solving the above equation. In numerical calculations the order m of cylindrical waves must be truncated. To ensure the satisfactory convergence of the eigenproblem, m is truncated by the condition $|D_m|>10^{-10}$.

We consider two lattices in the present work: the square and triangular lattices. For the square lattice, $\mathbf{a}_1=a(1,0)$ and $\mathbf{a}_2=a(0,1)$, where a is the so-called lattice constant. For the triangular lattice, $\mathbf{a}_1=a(1,0)$ and $\mathbf{a}_2=a(1/2,1/2\sqrt{3})$. Figure 1 shows our calculated relation between the wave numbers κ and the Bloch wave vectors, together with that obtained from a variational method [10]. Cylinders are arranged in the square lattice and the filling fraction, defined as the ratio of the area occupied by the cylinders in a primitive unit cell to that of the primitive unit cell, is 0.196. Obviously, our results based on the multiple scattering method are in good agreement with those based on the variational method. No complete band gaps exist. It is noted that if specific values of the lattice constant a and the liquid depth h are given, the conventional band structure ($\omega[K]$) can be derived.

Figure 2 shows the wave number κ versus wave vectors for the square and triangular lattices. For both lattices, the filling fraction is both 0.5. For the square lattice, there is a band gap in between the first and the second bands, for κ ranging from 1.005 to $1.287\pi/a$. The ratio of the gap width to the midgap value of κ is 25%. For the triangular lattice, there are two band gaps. The first band gap occurs in between the second and the third bands, while the second one is in between the sixth and the seventh bands. The band gaps span for κ from 1.902 to $2.040\pi/a$ and from 3.432 to $3.782\pi/a$ for the first and second band gaps, respectively. The ratio of the gap width to the midgap value of κ is 7% and 9.7% for the first and the second band gap, respectively. For frequencies within band gaps, propagation of liquid surface waves is forbidden, while for frequencies within bands, propagation is allowed.

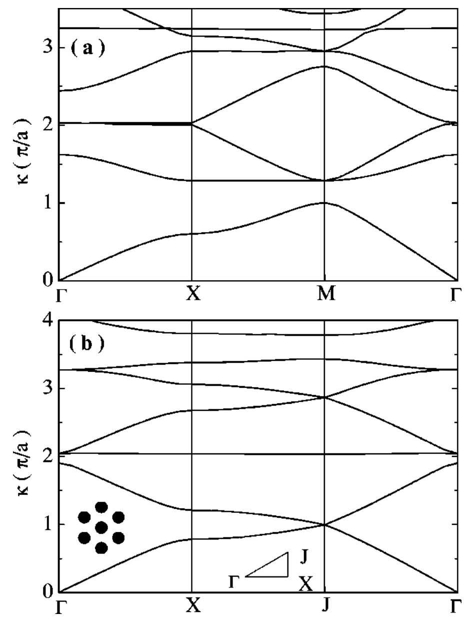

FIG. 2. Wave number κ versus wave vectors in the Brillouin zone along three highly symmetrical directions for the square (a) and triangular (b) lattices. The filling fraction is 0.5 for both cases. The geometry of cylinders (top view) and the irreducible Brillouin zone are shown as insets.

To show the influence of the filling fraction on the formation of band gaps, the gap map as a function of the filling fraction is shown in Fig. 3 for the square and triangular lattices. For the square lattices, there are two band gap regions. The first band gap occurs when the filling fraction is larger than 0.3. The band gap increases with the filling fraction, up to the filling fraction of 0.785, corresponding to the case that all cylinders touch each other. The second band gap occurs

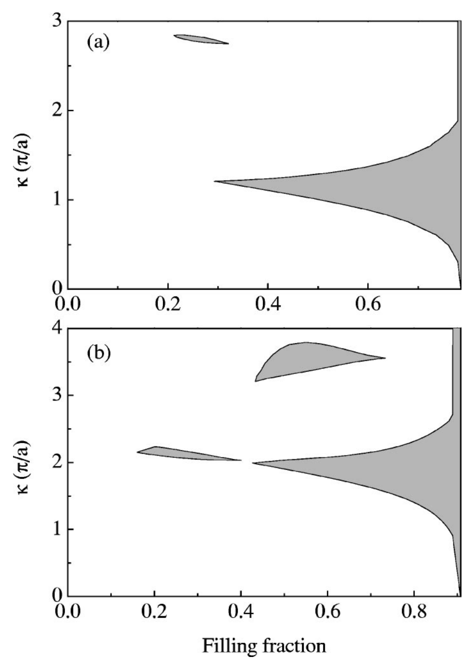

FIG. 3. Gap map as a function of the filling fraction for the square (a) and triangular (b) lattices. Gray area denotes band gap.

for the filling fraction between 0.21 and 0.32. For the triangular lattice, there are also two band gaps. The first band gap exists when the filling fraction is in the range between 0.16 and 0.4 or larger than 0.43. For the filling fraction larger than 0.43, the band gap increases with the filling fraction, up to the critical value of 0.907, corresponding to the case that all cylinders touch each other. The second band gap opens up when the filling fraction is larger than 0.43 and closes up when the filling fraction reaches 0.73. Clearly, there exist band gaps for liquid surface waves propagating through an array of cylinders in both lattices. In the previous study of water waves propagating through a square array of cylinders [10], no band gaps were found. The reason is that the filling fraction is outside of the regions in which there exist band gaps.

In the above discussions, the capillary effects are neglected. This approximation is valid when the wavelength of the surface wave is much larger than the capillary length. At very high frequencies or for cylinders very close to each other, capillary effects are important. The wetting of cylinders should be considered [29].

In summary, we calculated the band structures of liquid surface waves propagating through an array of vertical cylinders arranged in both the square and triangular lattices by the multiple scattering method. It is found that the filling fraction is an important parameter to determine the existence of band gaps. In both the square and triangular lattices band gaps were found.

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